

Basic properties of ultrafunctions

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Dedicated to Bernard Ruf in occasion of his 60th birthday.

Abstract

Ultrafunctions are a particular class of functions defined on a non-Archimedean field $\mathbb{R}^* \supset \mathbb{R}$. They provide generalized solutions to functional equations which do not have any solutions among the real functions or the distributions. In this paper we analyze systematically some basic properties of the spaces of ultrafunctions.

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1 Introduction

In some recent papers the notion of ultrafunction has been introduced ([1], [2]). Ultrafunctions are a particular class of functions defined on a non-Archimedean field $\mathbb{R}^* \supset \mathbb{R}$. We recall that a non-Archimedean field is an ordered field which contain infinite and infinitesimal numbers.

To any continuous function $f : \mathbb{R}^N \rightarrow \mathbb{R}$ we associate in a canonical way an ultrafunction $\tilde{f} : (\mathbb{R}^*)^N \rightarrow \mathbb{R}^*$ which extends f ; more exactly, to any functional vector space $V(\Omega) \subseteq L^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$, we associate a space of ultrafunctions $\tilde{V}(\Omega)$. The ultrafunctions are much more than the functions and among them we can find solutions of functional equations which do not have any solutions among the real functions or the distributions.

A typical example of this situation is analyzed in [2] where a simple Physical model is studied. In this problem there is a material point interacting with a field and, as it usually happens, the energy is infinite. Therefore the need to use infinite numbers arises naturally. Other situations in which infinite and infinitesimal numbers appear in a natural way are studied in [5] and in [6].

In this paper we analyze systematically some basic properties of the spaces of ultrafunctions $\tilde{V}(\Omega)$. In particular we will show that:

- to any measurable function f we can associate an unique ultrafunction \tilde{f} such that $f(x) = \tilde{f}(x)$ if f is continuous in a neighborhood of x ;
- to every distribution T we can associate an ultrafunction $\tilde{T}(x)$ such that $\forall \varphi \in \mathcal{D}, \langle T, \varphi \rangle = \int^* \tilde{T}(x) \tilde{\varphi}(x) dx$ where \int^* is a suitable extension of the integral to the ultrafunctions;
- the vector space of ultrafunctions $\tilde{V}(\Omega)$ is hyperfinite, namely it shares many properties of finite vector spaces (see Sec. 2.4);
- the vector space of ultrafunctions $\tilde{V}(\Omega)$ has an hyperfinite basis $\{\delta_a(x)\}_{a \in \Sigma}$ where δ_a is the "Dirac ultrafunction in a " (see Def. 18) and $\Sigma \subset (\mathbb{R}^*)^N$ is a suitable set;
- any ultrafunction u can be represented as follows:

$$u(x) = \sum_{q \in \Sigma} u(q) \sigma_q(x),$$

where $\{\sigma_a(x)\}_{a \in \Sigma}$ is the dual basis of $\{\delta_a(x)\}_{a \in \Sigma}$;

- any operator $F : V(\Omega) \rightarrow \mathcal{D}'(\Omega)$, can be extended to an operator

$$\tilde{F} : \tilde{V}(\Omega) \rightarrow \tilde{V}'(\Omega);$$

the extension of the derivative and the Fourier transform will be analyzed in some detail.

The techniques on which the notion of ultrafunction is based are related to Non Archimedean Mathematics (NAM) and to Nonstandard Analysis (NSA). The first section of this paper is devoted to a relatively elementary presentation of the basic notions of NAM and NSA inspired by [3] and [4]. Some technicalities have been avoided by presenting the matter in an axiomatic way. Of course, it is necessary to prove the consistency of the axioms. This is done in the appendix; however in the appendix we have assumed the reader to be familiar with NSA.

1.1 Notations

Let Ω be a subset of \mathbb{R}^N : then

- $\mathcal{C}(\Omega)$ denotes the set of continuous functions defined on $\Omega \subset \mathbb{R}^N$;
- $\mathcal{C}_0(\Omega)$ denotes the set of continuous functions in $\mathcal{C}(\Omega)$ having compact support in Ω ;
- $\mathcal{C}^k(\Omega)$ denotes the set of functions defined on $\Omega \subset \mathbb{R}^N$ which have continuous derivatives up to the order k ;
- $\mathcal{D}(\Omega)$ denotes the set of the infinitely differentiable functions with compact support defined on $\Omega \subset \mathbb{R}^N$; $\mathcal{D}'(\Omega)$ denotes the topological dual of $\mathcal{D}(\Omega)$, namely the set of distributions on Ω ;
- $H^{1,p}(\Omega)$ is the usual Sobolev space defined as the set of functions in $L^p(\Omega)$ such that $\nabla u \in L^p(\Omega)^N$;
- $H^1(\Omega) = H^{1,2}(\Omega)$
- for any $\xi \in (\mathbb{R}^N)^*$, $\rho \in \mathbb{R}^*$, we set $\mathfrak{B}_\rho(\xi) = \{x \in (\mathbb{R}^N)^* : |x - \xi| < \rho\}$;
- $\text{supp}(f) = \overline{\{x \in \mathbb{R}^N : f(x) \neq 0\}}$;
- $\text{mon}(x) = \{y \in \mathbb{R}^N : x \sim y\}$;
- $\text{gal}(x) = \{y \in \mathbb{R}^N : x \sim_f y\}$.

2 Λ -theory

In this section we present the basic notions of Non Archimedean Mathematics and of Nonstandard Analysis following a method inspired by [3] (see also [1] and [2]).

2.1 Non Archimedean Fields

Here, we recall the basic definitions and facts regarding non-Archimedean fields. In the following, \mathbb{K} will denote an ordered field. We recall that such a field contains (a copy of) the rational numbers. Its elements will be called numbers.

Definition 1 *Let \mathbb{K} be an ordered field. Let $\xi \in \mathbb{K}$. We say that:*

- ξ is infinitesimal if, for all positive $n \in \mathbb{N}$, $|\xi| < \frac{1}{n}$;
- ξ is finite if there exists $n \in \mathbb{N}$ such as $|\xi| < n$;
- ξ is infinite if, for all $n \in \mathbb{N}$, $|\xi| > n$ (equivalently, if ξ is not finite).

Definition 2 *An ordered field \mathbb{K} is called Non-Archimedean if it contains an infinitesimal $\xi \neq 0$.*

It's easily seen that all infinitesimal are finite, that the inverse of an infinite number is a nonzero infinitesimal number, and that the inverse of a nonzero infinitesimal number is infinite.

Definition 3 *A superreal field is an ordered field \mathbb{K} that properly extends \mathbb{R} .*

It is easy to show, due to the completeness of \mathbb{R} , that there are nonzero infinitesimal numbers and infinite numbers in any superreal field. Infinitesimal numbers can be used to formalize a new notion of "closeness":

Definition 4 *We say that two numbers $\xi, \zeta \in \mathbb{K}$ are infinitely close if $\xi - \zeta$ is infinitesimal. In this case, we write $\xi \sim \zeta$.*

Clearly, the relation " \sim " of infinite closeness is an equivalence relation.

Theorem 5 *If \mathbb{K} is a superreal field, every finite number $\xi \in \mathbb{K}$ is infinitely close to a unique real number $r \sim \xi$, called the **shadow** or the **standard part** of ξ .*

Given a finite number ξ , we denote its shadow as $sh(\xi)$, and we put $sh(\xi) = +\infty$ ($sh(\xi) = -\infty$) if $\xi \in \mathbb{K}$ is a positive (negative) infinite number.

Definition 6 *Let \mathbb{K} be a superreal field, and $\xi \in \mathbb{K}$ a number. The monad of ξ is the set of all numbers that are infinitely close to it:*

$$\mathbf{mon}(\xi) = \{\zeta \in \mathbb{K} : \xi \sim \zeta\},$$

and the galaxy of ξ is the set of all numbers that are finitely close to it:

$$\mathbf{gal}(\xi) = \{\zeta \in \mathbb{K} : \xi - \zeta \text{ is finite}\}$$

By definition, it follows that the set of infinitesimal numbers is $\mathbf{mon}(0)$ and that the set of finite numbers is $\mathbf{gal}(0)$.

2.2 The Λ -limit

In this section we will introduce a superreal field \mathbb{K} and we will analyze its main properties by mean of the Λ -theory (see also [1], [2]).

\mathbb{U} will denote our "mathematical universe". For our applications a good choice of \mathbb{U} is given by the superstructure on \mathbb{R} :

$$\mathbb{U} = \bigcup_{n=0}^{\infty} \mathbb{U}_n$$

where \mathbb{U}_n is defined by induction as follows:

$$\begin{aligned} \mathbb{U}_0 &= \mathbb{R}; \\ \mathbb{U}_{n+1} &= \mathbb{U}_n \cup \mathcal{P}(\mathbb{U}_n). \end{aligned}$$

Here $\mathcal{P}(E)$ denotes the power set of E . Identifying the couples with the Kuratowski pairs and the functions and the relations with their graphs, it follows that \mathbb{U} contains almost every usual mathematical object. Given the universe \mathbb{U} , we denote by \mathcal{F} the family of finite subsets of \mathbb{U} . Clearly (\mathcal{F}, \subset) is a directed set and, as usual, a function $\varphi : \mathcal{F} \rightarrow E$ will be called *net* (with values in E). We present axiomatically the notion of Λ -limit:

Axioms of the Λ -limit

- **(Λ -1) Existence Axiom.** *There is a superreal field $\mathbb{K} \supset \mathbb{R}$ such that every net $\varphi : \mathcal{F} \rightarrow \mathbb{R}$ has a unique limit $L \in \mathbb{K}$ (called the " Λ -limit" of φ .) The Λ -limit of φ will be denoted as*

$$L = \lim_{\lambda \uparrow \mathbb{U}} \varphi(\lambda).$$

Moreover we assume that every $\xi \in \mathbb{K}$ is the Λ -limit of some real function $\varphi : \mathcal{F} \rightarrow \mathbb{R}$.

- **(Λ -2) Real numbers axiom.** *If $\varphi(\lambda)$ is eventually constant, namely $\exists \lambda_0 \in \mathcal{F}, r \in \mathbb{R}$ such that $\forall \lambda \supset \lambda_0, \varphi(\lambda) = r$, then*

$$\lim_{\lambda \uparrow \mathbb{U}} \varphi(\lambda) = r.$$

- **(Λ -3) Sum and product Axiom.** *For all $\varphi, \psi : \mathcal{F} \rightarrow \mathbb{R}$:*

$$\begin{aligned} \lim_{\lambda \uparrow \mathbb{U}} \varphi(\lambda) + \lim_{\lambda \uparrow \mathbb{U}} \psi(\lambda) &= \lim_{\lambda \uparrow \mathbb{U}} (\varphi(\lambda) + \psi(\lambda)); \\ \lim_{\lambda \uparrow \mathbb{U}} \varphi(\lambda) \cdot \lim_{\lambda \uparrow \mathbb{U}} \psi(\lambda) &= \lim_{\lambda \uparrow \mathbb{U}} (\varphi(\lambda) \cdot \psi(\lambda)). \end{aligned}$$

Theorem 7 *The set of axioms $\{(\Lambda-1), (\Lambda-2), (\Lambda-3)\}$ is consistent.*

Theorem 7 will be proved in the Appendix.

Now we want to define the Λ -limit of any bounded net of mathematical objects in \mathbb{U} (a net $\varphi : \mathcal{F} \rightarrow \mathbb{U}$ is called bounded if there exists n such that $\forall \lambda \in \mathcal{F}, \varphi(\lambda) \in \mathbb{U}_n$). To this aim, consider a net

$$\varphi : \mathcal{F} \rightarrow \mathbb{U}_n. \quad (1)$$

We will define $\lim_{\lambda \uparrow \mathbb{U}} \varphi(\lambda)$ by induction on n . For $n = 0$, $\lim_{\lambda \uparrow \mathbb{U}} \varphi(\lambda)$ is defined by the axioms $(\Lambda-1), (\Lambda-2), (\Lambda-3)$; so by induction we may assume that the limit is defined for $n - 1$ and we define it for the net (1) as follows:

$$\lim_{\lambda \uparrow \mathbb{U}} \varphi(\lambda) = \left\{ \lim_{\lambda \uparrow \mathbb{U}} \psi(\lambda) \mid \psi : \mathcal{F} \rightarrow \mathbb{U}_{n-1} \text{ and } \forall \lambda \in \mathcal{F}, \psi(\lambda) \in \varphi(\lambda) \right\}.$$

Definition 8 *A mathematical entity (number, set, function or relation) which is the Λ -limit of a net is called **internal**.*

2.3 Natural extensions of sets and functions

Definition 9 *The **natural extension** of a set $E \subset \mathbb{R}$ is given by*

$$E^* := \lim_{\lambda \uparrow \mathbb{U}} c_E(\lambda) = \left\{ \lim_{\lambda \uparrow \mathbb{U}} \psi(\lambda) \mid \psi(\lambda) \in E \right\}$$

where $c_E(\lambda)$ is the net identically equal to E .

This definition, combined with axiom $(\Lambda-1)$, entails that

$$\mathbb{K} = \mathbb{R}^*.$$

In this context a function f can be identified with its graph; then the natural extension of a function is well defined. Moreover we have the following result:

Theorem 10 *The **natural extension** of a function*

$$f : E \rightarrow F$$

is a function

$$f^* : E^* \rightarrow F^*$$

and for every net $\varphi : \mathcal{F} \cap \mathcal{P}(E) \rightarrow E$, and every function $f : E \rightarrow F$, we have that

$$\lim_{\lambda \uparrow \mathbb{U}} f(\varphi(\lambda)) = f^* \left(\lim_{\lambda \uparrow \mathbb{U}} \varphi(\lambda) \right).$$

When dealing with functions, sometimes the "*" will be omitted if the domain of the function is clear from the context. For example, if $\eta \in \mathbb{R}^*$ is an infinitesimal, then clearly e^η denotes $\exp^*(\eta)$.

The following theorem is a fundamental tool in using the Λ -limit:

Theorem 11 (Leibnitz Principle) Let \mathcal{R} be a relation in \mathbb{U}_n for some $n \geq 0$ and let $\varphi, \psi : \mathcal{F} \rightarrow \mathbb{U}_n$. If

$$\forall \lambda \in \mathcal{F}, \varphi(\lambda) \mathcal{R} \psi(\lambda)$$

then

$$\left(\lim_{\lambda \uparrow \mathbb{U}} \varphi(\lambda) \right) \mathcal{R}^* \left(\lim_{\lambda \uparrow \mathbb{U}} \psi(\lambda) \right).$$

When \mathcal{R} is \in or $=$ we will not use the symbol $*$ to denote their extensions, since their meaning is unaltered in \mathbb{R}^* .

2.4 Hyperfinite extensions

Definition 12 An internal set is called **hyperfinite** if it is the Λ -limit of a net $\varphi : \mathcal{F} \rightarrow \mathcal{F}$.

Definition 13 Given any set $E \in \mathbb{U}$, the hyperfinite extension of E is defined as follows:

$$E^\circ := \lim_{\lambda \uparrow \mathbb{U}} (E \cap \lambda).$$

All the internal finite sets are hyperfinite, but there are hyperfinite sets which are not finite. For example the set

$$\mathbb{R}^\circ := \lim_{\lambda \uparrow \mathbb{U}} (\mathbb{R} \cap \lambda)$$

is not finite. The hyperfinite sets are very important since they inherit many properties of finite sets via Leibnitz principle. For example, \mathbb{R}° has the maximum and the minimum and every internal function

$$f : \mathbb{R}^\circ \rightarrow \mathbb{R}^*$$

has the maximum and the minimum as well.

Also, it is possible to add the elements of an hyperfinite set of numbers or vectors as follows: let

$$A := \lim_{\lambda \uparrow \mathbb{U}} A_\lambda$$

be an hyperfinite set; then the hyperfinite sum is defined in the following way:

$$\sum_{a \in A} a = \lim_{\lambda \uparrow \mathbb{U}} \sum_{a \in A_\lambda} a.$$

In particular, if $A_\lambda = \{a_1(\lambda), \dots, a_{\beta(\lambda)}(\lambda)\}$ with $\beta(\lambda) \in \mathbb{N}$, then setting

$$\beta = \lim_{\lambda \uparrow \mathbb{U}} \beta(\lambda) \in \mathbb{N}^*$$

we use the notation

$$\sum_{j=1}^{\beta} a_j = \lim_{\lambda \uparrow \mathbb{U}} \sum_{j=1}^{\beta(\lambda)} a_j(\lambda).$$

2.5 Qualified sets

When we have a net $\varphi : Q \rightarrow \mathbb{U}_n$, where $Q \subset \mathcal{F}$, we can define the Λ -limit of φ by posing

$$\lim_{\lambda \in Q} \varphi(\lambda) = \lim_{\lambda \uparrow \mathbb{U}} \tilde{\varphi}(\lambda)$$

where

$$\tilde{\varphi}(\lambda) = \begin{cases} \varphi(\lambda) & \text{for } \lambda \in Q \\ \emptyset & \text{for } \lambda \notin Q \end{cases}$$

As one can expect, if two nets φ, ψ are equal on a "large" or a "qualified" subset of \mathcal{F} then they share the same Λ -limit. The notion of "qualified" subset of \mathcal{F} can be precisely defined as follows:

Definition 14 *We say that a set $Q \subset \mathcal{F}$ is qualified if for every bounded net φ we have that*

$$\lim_{\lambda \uparrow \mathbb{U}} \varphi(\lambda) = \lim_{\lambda \in Q} \varphi(\lambda).$$

By the above definition, we have that the Λ -limit of a net φ depends only on the values that φ takes on a qualified set (it is in this sense that we could imagine Q to be "large"). It is easy to see that (nontrivial) qualified sets exist. For example by (A-2) we deduce that, for every $\lambda_0 \in \mathcal{F}$, the set

$$Q(\lambda_0) := \{\lambda \in \mathcal{F} \mid \lambda_0 \subseteq \lambda\}$$

is qualified. In this paper, we will use the notion of qualified set via the following Theorem:

Theorem 15 *Let \mathcal{R} be a relation in \mathbb{U}_n for some $n \geq 0$ and let $\varphi, \psi : \mathcal{F} \rightarrow \mathbb{U}_n$. Then the following statements are equivalent:*

- *there exists a qualified set Q such that*

$$\forall \lambda \in Q, \varphi(\lambda) \mathcal{R} \psi(\lambda);$$

- *we have*

$$\left(\lim_{\lambda \uparrow \mathbb{U}} \varphi(\lambda) \right) \mathcal{R}^* \left(\lim_{\lambda \uparrow \mathbb{U}} \psi(\lambda) \right).$$

Proof: It is an immediate consequence of Theorem 11 and the definition of qualified set.

□

3 Ultrafunctions

In this section, we will introduce the notion of ultrafunction and we will analyze its first properties.

3.1 Definition of Ultrafunctions

Let Ω be a set in \mathbb{R}^N , and let $V(\Omega)$ be a (real or complex) vector space such that $\mathcal{D}(\overline{\Omega}) \subseteq V(\Omega) \subseteq L^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$.

Definition 16 *Given the function space $V(\Omega)$ we set*

$$\tilde{V}(\Omega) := \lim_{\lambda \uparrow \mathbb{U}} V_\lambda(\Omega) = \text{Span}^*(V(\Omega)^\circ),$$

where

$$V_\lambda(\Omega) = \text{Span}(V(\Omega) \cap \lambda).$$

$\tilde{V}(\Omega)$ will be called the **space of ultrafunctions** generated by $V(\Omega)$.

So, given any vector space of functions $V(\Omega)$, the space of ultrafunction generated by $V(\Omega)$ is a vector space of hyperfinite dimension that includes $V(\Omega)$, and the ultrafunctions are Λ -limits of functions in V_λ . Hence the ultrafunctions are particular internal functions

$$u : (\mathbb{R}^*)^N \rightarrow \mathbb{C}^*.$$

Observe that, by definition, the dimension of $\tilde{V}(\Omega)$ (that we denote by β) is equal to the internal cardinality of any of its bases, and the following formula holds:

$$\beta = \lim_{\lambda \uparrow \mathbb{U}} \dim(V_\lambda(\Omega)).$$

Since $\tilde{V}(\Omega) \subset [L^2(\mathbb{R})]^*$, it can be equipped with the following scalar product

$$(u, v) = \int^* u(x) \overline{v(x)} \, dx,$$

where \int^* is the natural extension of the Lebesgue integral considered as a functional

$$\int : L^1(\Omega) \rightarrow \mathbb{C}.$$

Notice that the Euclidean structure of $\tilde{V}(\Omega)$ is the Λ -limit of the Euclidean structure of every V_λ given by the usual L^2 scalar product. The norm of an ultrafunction will be given by

$$\|u\| = \left(\int^* |u(x)|^2 \, dx \right)^{\frac{1}{2}}.$$

Remark 17 *Notice that the natural extension f^* of a function f is an ultrafunction if and only if $f \in V(\Omega)$.*

Proof. Let $f \in V(\Omega)$, and $Q(f) = \{\lambda \in \mathcal{F} \mid f \in \lambda\}$. Since, for every $\lambda \in Q(f)$, $f \in V_\lambda(\Omega)$ and, as we observed in section 2.3, $Q(f)$ is a qualified set, it follows by Theorem 15 that $f^* \in \tilde{V}(\Omega)$.

Conversely, if $f \notin V(\Omega)$ then by Leibnitz Principle it follows that $f^* \notin V^*(\Omega)$ and, since $\tilde{V}(\Omega) \subset V^*(\Omega)$, this entails the thesis.

□

3.2 Delta, Sigma and Theta Basis

In this section we introduce three particular kinds of bases for $V(\Omega)$ and we study their main properties. We start by defining the *Delta ultrafunctions*:

Definition 18 *Given a number $q \in \Omega^*$, we denote by $\delta_q(x)$ an ultrafunction in $\tilde{V}(\Omega)$ such that*

$$\forall v \in \tilde{V}(\Omega), \int^* v(x) \delta_q(x) dx = v(q). \quad (2)$$

$\delta_q(x)$ is called *Delta (or the Dirac) ultrafunction concentrated in q* .

Let us see the main properties of the Delta ultrafunctions:

Theorem 19 *We have the following properties:*

1. *For every $q \in \Omega^*$ there exists an unique Delta ultrafunction concentrated in q ;*
2. *for every $a, b \in \Omega^*$ $\delta_a(b) = \delta_b(a)$;*
3. *$\|\delta_q\|^2 = \delta_q(q)$.*

Proof. 1. Let $\{e_j\}_{j=1}^\beta$ be an orthonormal real basis of $\tilde{V}(\Omega)$, and set

$$\delta_q(x) = \sum_{j=1}^\beta e_j(q) e_j(x).$$

Let us prove that $\delta_q(x)$ actually satisfies (2). Let $v(x) = \sum_{j=1}^\beta v_j e_j(x)$ be any ultrafunction. Then

$$\begin{aligned} \int^* v(x) \delta_q(x) dx &= \int^* \left(\sum_{j=1}^\beta v_j e_j(x) \right) \left(\sum_{k=1}^\beta e_k(q) e_k(x) \right) dx = \\ &= \sum_{j=1}^\beta \sum_{k=1}^\beta v_j e_k(q) \int^* e_j(x) e_k(x) dx = \\ &= \sum_{j=1}^\beta \sum_{k=1}^\beta v_j e_k(q) \delta_{j,q} = \sum_{j=1}^\beta v_j e_j(q) = v(q). \end{aligned}$$

So $\delta_q(x)$ is a Delta ultrafunction centered in q .

It is unique: if $f_q(x)$ is another Delta ultrafunction centered in q then for every $y \in \Omega^*$ we have:

$$\delta_q(y) - f_q(y) = \int^* (\delta_q(x) - f_q(x)) \delta_y(x) dx = \delta_y(q) - \delta_y(q) = 0$$

and hence $\delta_q(y) = f_q(y)$ for every $y \in \Omega^*$.

2. $\delta_a(b) = \int^* \delta_a(x) \delta_b(x) dx = \delta_b(a).$
3. $\|\delta_q\|^2 = \int^* \delta_q(x) \delta_q(x) = \delta_q(q).$

□

Definition 20 A Delta-basis $\{\delta_a(x)\}_{a \in \Sigma}$ ($\Sigma \subset \Omega^*$) is a basis for $\tilde{V}(\Omega)$ whose elements are Delta ultrafunctions. Its dual basis $\{\sigma_a(x)\}_{a \in \Sigma}$ is called Sigma-basis. We recall that, by definition of dual basis, for every $a, b \in \Omega^*$ the equation

$$\int^* \delta_a(x) \sigma_b(x) dx = \delta_{ab} \quad (3)$$

holds. The set $\Sigma \subset \Omega^*$ is called set of independent points.

The existence of a Delta-basis is an immediate consequence of the following fact:

Remark 21 The set $\{\delta_a(x) | a \in \Omega^*\}$ generates all $\tilde{V}(\Omega)$. In fact, let $G(\Omega)$ be the vectorial space generated by the set $\{\delta_a(x) | a \in \Omega^*\}$ and suppose that $G(\Omega)$ is properly included in $\tilde{V}(\Omega)$. Then the orthogonal $G(\Omega)^\perp$ of $G(\Omega)$ in $\tilde{V}(\Omega)$ contains a function $f \neq 0$. But, since $f \in G(\Omega)^\perp$, for every $a \in \Omega^*$ we have

$$f(a) = \int^* f(x) \delta_a(x) dx = 0,$$

so $f|_{\Omega^*} = 0$ and this is absurd. Thus the set $\{\delta_a(x) | a \in \Omega^*\}$ generates $\tilde{V}(\Omega)$, hence it contains a basis.

Let us see some properties of Delta- and Sigma-bases:

Theorem 22 A Delta-basis $\{\delta_q(x)\}_{q \in \Sigma}$ and its dual basis $\{\sigma_q(x)\}_{q \in \Sigma}$ satisfy the following properties:

1. if $u \in \tilde{V}(\Omega)$, then

$$u(x) = \sum_{q \in \Sigma} \left(\int^* \sigma_q(\xi) u(\xi) d\xi \right) \delta_q(x);$$

2. if $u \in \tilde{V}(\Omega)$, then

$$u(x) = \sum_{q \in \Sigma} u(q) \sigma_q(x); \quad (4)$$

3. if two ultrafunctions u and v coincide on a set of independent points then they are equal;
4. if Σ is a set of independent points and $a, b \in \Sigma$ then $\sigma_a(b) = \delta_{ab}$;
5. for any $q \in \Omega^*$, $\sigma_q(x)$ is well defined.

Proof. 1. It is an immediate consequence of the definition of dual basis.
2. Since $\{\delta_q(x)\}_{q \in \Sigma}$ is the dual basis of $\{\sigma_q(x)\}_{q \in \Sigma}$ we have that

$$u(x) = \sum_{q \in \Sigma} \left(\int \delta_q(\xi) u(\xi) d\xi \right) \sigma_q(x) = \sum_{q \in \Sigma} u(q) \sigma_q(x).$$

3. It follows directly from 2.

4. It follows directly by equation (3)

5. Given any point $q \in \Omega^*$ clearly there is a Delta-basis $\{\delta_a(x)\}_{a \in \Sigma}$ with $q \in \Sigma$. Then $\sigma_q(x)$ can be defined by mean of the basis $\{\delta_a(x)\}_{a \in \Sigma}$. We have to prove that, given another Delta basis $\{\delta_a(x)\}_{a \in \Sigma'}$ with $q \in \Sigma'$, the corresponding $\sigma'_q(x)$ is equal to $\sigma_q(x)$. Using (2), with $u(x) = \sigma'_q(x)$, we have that

$$\sigma'_q(x) = \sum_{a \in \Sigma} \sigma'_q(a) \sigma_a(x).$$

Then, by (4), it follows that $\sigma'_q(x) = \sigma_q(x)$.

□

Let Σ be a set of independent points, and let $L_\Sigma : \tilde{V}(\Omega) \rightarrow \tilde{V}(\Omega)$ be the linear operator such that

$$L_\Sigma \sigma_a(x) = \delta_a(x)$$

for every $a \in \Sigma$.

Proposition 23 L_Σ is selfadjoint, positive and

$$\int^* L_\Sigma u(x) v(x) dx = \sum_{a \in \Sigma} u(a) v(a).$$

Proof. Since $u(x) = \sum_{a \in \Sigma} u(a) \sigma_a(x)$ and $v(x) = \sum_{a \in \Sigma} v(a) \sigma_a(x)$, then

$$\begin{aligned} \int^* L_\Sigma u(x) v(x) dx &= \int^* L_\Sigma \left(\sum_{a \in \Sigma} u(a) \sigma_a(x) \right) \left(\sum_{b \in \Sigma} v(b) \sigma_b(x) \right) dx = \\ &= \sum_{a \in \Sigma} \sum_{b \in \Sigma} u(a) v(b) \int^* \delta_a(x) \sigma_b(x) dx = \sum_{a \in \Sigma} u(a) v(a). \end{aligned}$$

Hence, clearly, L_Σ is selfadjoint and positive.

□

From now on, we consider the set Σ fixed once for all and we simply denote the operator L_Σ by L . Since L is a positive selfadjoint operator, $A = L^{1/2}$ is a well defined positive selfadjoint operator. For every $a \in \Sigma$ we set

$$\theta_a(x) = A \sigma_a(x).$$

Theorem 24 The following properties hold:

1. $\{\theta_a(x)\}_{a \in \Sigma}$ is an orthonormal basis;

2. for every $a, b \in \Sigma$, $\theta_a(b) = \theta_b(a)$;

3. for every ultrafunction u we have

$$u(x) = \sum_{a \in \Sigma} u(a) \sigma_a(x) = \sum_{a \in \Sigma} \underline{u}(a) \theta_a(x) = \sum_{a \in \Sigma} \underline{\underline{u}}(a) \delta_a(x),$$

where we have set, for every $a \in \Sigma$,

$$\underline{u}(a) := (A^{-1}u)(a) = \int^* \theta_a(\xi) u(\xi) d\xi;$$

$$\underline{\underline{u}}(a) = (A^{-1}\underline{u})(a) = (L^{-1}u)(a) = \int^* \sigma_a(\xi) u(\xi) d\xi;$$

4. for every ultrafunctions u, v we have

$$\int^* u(x) v(x) dx = \sum_{a \in \Sigma} \underline{u}(a) \underline{v}(a) = \sum_{a \in \Sigma} \underline{\underline{u}}(a) v(a);$$

5. for every ultrafunction u we have

$$\int^* u(x) dx = \sum_{a \in \Sigma} \underline{\underline{u}}(a).$$

Proof: 1) $\{\theta_a(x)\}_{a \in \Sigma}$ is a basis since it is the image of the basis $\{\sigma_a(x)\}_{a \in \Sigma}$ respect to the invertible linear application L . It is orthonormal: for every $a, b \in \Sigma$ we have

$$\begin{aligned} \int^* \theta_a(x) \theta_b(x) dx &= \int^* A\sigma_a(x) A\sigma_b(x) dx = \int^* L\sigma_a(x) \sigma_b(x) dx = \\ &= \sigma_b(a) = \delta_{ab}. \end{aligned}$$

2) We have

$$\begin{aligned} \theta_a(b) &= \int^* \theta_a(x) \delta_b(x) dx = \int^* \theta_a(x) A\theta_b(x) dx = \\ &= \int^* A\theta_a(x) \theta_b(x) dx = \int^* \delta_a(x) \theta_b(x) dx = \theta_b(a). \end{aligned}$$

3) The equality

$$u(x) = \sum_{a \in \Sigma} u(a) \sigma_a(x)$$

has been proved in Theorem 22, (4); the equality

$$u(x) = \sum_{a \in \Sigma} \underline{u}(a) \theta_a(x),$$

where $\underline{u}(a) = \int^* \theta_a(\xi) u(\xi) d\xi$, follows since $\{\theta_a(x)\}_{a \in \Sigma}$ is an orthonormal basis. And

$$\begin{aligned} (A^{-1}u)(a) &= \int^* \delta_a(\xi) A^{-1}u(\xi) d\xi = \\ &= \int^* A^{-1} \delta_a(\xi) u(\xi) d\xi = \int^* \theta_a(\xi) u(\xi) d\xi \end{aligned}$$

since A (and, so, A^{-1}) is selfadjoint.

The equality

$$u(x) = \sum_{a \in \Sigma} \underline{u}(a) \delta_a(x),$$

where $\underline{u}(a) = \int^* \sigma_a(\xi) u(\xi) d\xi$, follows by point (1) in Theorem 22. And

$$\begin{aligned} \underline{u}(a) &= \int^* \sigma_a(\xi) u(\xi) d\xi = \int^* L^{-1} \delta_a(\xi) u(\xi) d\xi = \\ &= \int^* \delta_a(\xi) L^{-1} u(\xi) d\xi = (L^{-1}u)(a). \end{aligned}$$

4) We have that $\int^* u(x) v(x) dx = \sum_{a \in \Sigma} \underline{u}(a) \underline{v}(a)$ since $\{\theta_a(x)\}_{a \in \Sigma}$ is an orthonormal basis:

$$\begin{aligned} \int^* u(x) v(x) dx &= \int^* \left(\sum_{a \in \Sigma} \underline{u}(a) \theta_a(x) \right) \left(\sum_{b \in \Sigma} \underline{v}(b) \theta_b(x) \right) dx = \\ &= \sum_{a \in \Sigma} \sum_{b \in \Sigma} \underline{u}(a) \underline{v}(b) \int^* \theta_a(x) \theta_b(x) dx = \sum_{a \in \Sigma} \underline{u}(a) \underline{v}(a); \end{aligned}$$

the equality $\int^* u(x) v(x) dx = \sum_{a \in \Sigma} \underline{u}(a) \underline{v}(a)$ follows by expressing $u(x)$ in the Delta basis and $v(x)$ in the Sigma basis:

$$\begin{aligned} \int^* u(x) v(x) dx &= \int^* \left(\sum_{a \in \Sigma} \underline{u}(a) \delta_a(x) \right) \left(\sum_{b \in \Sigma} v(b) \sigma_b(x) \right) dx = \\ &= \sum_{a \in \Sigma} \sum_{b \in \Sigma} v(b) \underline{u}(a) \int^* \delta_a(x) \sigma_b(x) dx = \sum_{a \in \Sigma} \underline{u}(a) \underline{v}(a). \end{aligned}$$

5) This follows by expressing $u(x)$ in the Delta basis:

$$\int^* u(x) dx = \int^* \sum_{a \in \Sigma} \underline{u}(a) \delta_a(x) dx = \sum_{a \in \Sigma} \underline{u}(a) \int^* \delta_a(x) dx = \sum_{a \in \Sigma} \underline{u}(a).$$

□

3.3 Canonical extension of a function

Let $V'(\Omega)$ denote the dual of $V(\Omega)$ and let \mathfrak{M} denote the set of measurable functions in \mathbb{R}^N . If $T \in V'(\Omega)$ and if there is a function $f \in \mathfrak{M}$ such that

$$\forall v \in V(\Omega), \langle T, v \rangle = \int f(x)v(x)dx$$

then T and f will be identified, and with some abuse of notation we shall write $T = f \in V'(\Omega) \cap \mathfrak{M}$. With this identification, $V'(\Omega) \cap \mathfrak{M} \subset L^2$.

Definition 25 *If $T \in [V'(\Omega)]^*$, there exists a unique ultrafunction $\widetilde{T}(x)$ such that*

$$\forall v \in \widetilde{V}(\Omega), \langle T, v \rangle = \int^* \widetilde{T}(x)v(x)dx.$$

In particular, if $u \in [V'(\Omega) \cap \mathfrak{M}]^$, \widetilde{u} will denote the unique ultrafunction such that*

$$\forall v \in \widetilde{V}(\Omega), \int^* u(x)v(x)dx = \int^* \widetilde{u}(x)v(x)dx.$$

Notice that $V'(\Omega) \cap \mathfrak{M}$ is a space of distributions which contains the delta measures, so to every Delta distribution δ_q is associated an ultrafunction which, by definition, is the Delta ultrafunction centered in q , as expected.

Definition 26 *If $f \in V'(\Omega) \cap \mathfrak{M}$, (f^*) is called the canonical extension of f . In the following, since f and f^* can be identified, we will write \widetilde{f} instead of (f^*) .*

Thus any function

$$f : \mathbb{R}^N \rightarrow \mathbb{R}$$

can be extended to the function

$$f^* : (\mathbb{R}^*)^N \rightarrow \mathbb{R}^*$$

which is called the natural extension of f and if $f \in V'(\Omega) \cap \mathfrak{M}$, we have also the canonical extension of f given by

$$\widetilde{f} : (\mathbb{R}^*)^N \rightarrow \mathbb{R}^*$$

If $f \notin V(\Omega)$, by Remark 17, $\widetilde{f} \neq f^*$, thus $f^* \notin \widetilde{V}(\Omega)$.

Example: if $\Omega = (-1, 1)$, then $|x|^{-1/2} \in V(-1, 1)' \cap \mathfrak{M}$; the ultrafunction $\widetilde{|x|^{-1/2}}$ is different from $(|x|^{-1/2})^*$ since the latter is not defined for $x = 0$, while

$$\left(\widetilde{|x|^{-1/2}} \right)_{x=0} = \int^* |x|^{-1/2} \delta_0(x) dx.$$

Theorem 27 *If $T \in [V(\Omega)']^*$, then*

$$\begin{aligned}\tilde{T}(x) &= \sum_{q \in \Sigma} \langle T, \delta_q \rangle \sigma_q(x) = \\ &= \sum_{q \in \Sigma} \langle T, \theta_q \rangle \theta_q(x) = \\ &= \sum_{q \in \Sigma} \langle T, \sigma_q \rangle \delta_q(x).\end{aligned}$$

In particular, if $f \in [V'(\Omega) \cap \mathfrak{M}]^$*

$$\tilde{f}(x) = \sum_{q \in \Sigma} \left[\int f^*(\xi) \delta_q(\xi) d\xi \right] \sigma_q(x) = \quad (5)$$

$$= \sum_{q \in \Sigma} \left[\int f^*(\xi) \theta_q(\xi) d\xi \right] \theta_q(x) = \quad (6)$$

$$= \sum_{q \in \Sigma} \left[\int f^*(\xi) \sigma_q(\xi) d\xi \right] \delta_q(x). \quad (7)$$

Proof. It is sufficient to prove that

$$\forall v \in V(\Omega), \quad \int \sum_{q \in \Sigma} \langle T, \delta_q \rangle \sigma_q(x) v(x) dx = \langle T, v \rangle.$$

We have that

$$\begin{aligned}\int \sum_{q \in \Sigma} \langle T, \delta_q \rangle \sigma_q(x) v(x) dx &= \sum_{q \in \Sigma} \langle T, \delta_q \rangle \int \sigma_q(x) v(x) dx = \\ &= \left\langle T, \sum_{q \in \Sigma} \left(\int \sigma_q(x) v(x) dx \right) \delta_q \right\rangle = \langle T, v \rangle.\end{aligned}$$

The other equalities can be proved similarly.

□

3.4 Ultrafunctions and distributions

In this section we will show that the space of ultrafunctions is richer than the space of distribution, in the sense that any distribution can be represented by an ultrafunction and that the converse is not true.

Definition 28 *Let $D \subset \tilde{V}(\Omega)$ be a vector space. We say that two ultrafunctions u and v are D -equivalent if*

$$\forall \varphi \in D, \quad \int^* (u(x) - v(x)) \varphi(x) dx = 0.$$

We say that two ultrafunctions u and v are distributionally equivalent if they are $\mathcal{D}(\Omega)$ -equivalent.

Theorem 29 *Given $T \in \mathcal{D}'$, there exists an ultrafunction u such that*

$$\forall \varphi \in \mathcal{D}(\Omega), \quad \int^* u(x) \varphi^*(x) dx = \langle T, \varphi \rangle. \quad (8)$$

Proof: Let $\{e_j(x)\}_{j \in J}$ be an orthonormal basis of the hyperfinite space $\tilde{V}(\Omega) \cap \mathcal{D}(\Omega)^*$ and take

$$u(x) = \sum_{j \in J} \langle T^*, e_j \rangle e_j(x).$$

Now take $\varphi \in \mathcal{D}$. Since $\varphi^* \in \tilde{V}(\Omega) \cap \mathcal{D}(\Omega)^*$, we have that

$$\varphi^*(x) = \sum_{j \in J} \left(\int^* \varphi^*(\xi) e_j(\xi) d\xi \right) e_j(x).$$

Thus

$$\begin{aligned} \int^* u(x) \varphi^*(x) dx &= \int^* \sum_{j \in J} \langle T^*, e_j \rangle e_j(x) \varphi^*(x) dx = \sum_{j \in J} \left\langle T^*, e_j \int^* e_j(x) \varphi^*(x) dx \right\rangle = \\ &= \left\langle T^*, \sum_{j \in J} \left(\int^* e_j(x) \varphi^*(x) dx \right) e_j \right\rangle = \langle T^*, \varphi^* \rangle = \langle T, \varphi \rangle. \end{aligned}$$

□

The following proposition shows that the ultrafunction u associated to the distribution T by (8) is not unique:

Proposition 30 *Take $T \in \mathcal{D}'(\Omega)$ and let*

$$V_T = \{u \in \tilde{V}(\Omega) : \forall \varphi \in \mathcal{D}(\Omega), \quad \int^* u(x) \varphi^*(x) dx = \langle T, \varphi \rangle\},$$

let $u \in V_T$ and let v be any ultrafunction. Then

1. $v \in V_T$ if and only if u and v are $\mathcal{D}(\Omega)$ -equivalent;
2. V_T is infinite.

Proof: 1) If $v \in V_T$ then $\forall \varphi \in \mathcal{D}(\Omega)$, $\int^* (u(x) - v(x)) \varphi^*(x) dx = \langle T, \varphi \rangle - \langle T, \varphi \rangle = 0$, so u and v are $\mathcal{D}(\Omega)$ -equivalent; conversely, if u and v are \mathcal{D} -equivalent then $\forall \varphi \in \mathcal{D}(\Omega)$, $\int^* u(x) \varphi^*(x) dx = \int^* v(x) \varphi^*(x) dx$. Since $\int^* u(x) \varphi^*(x) dx = \langle T, \varphi \rangle$ then $v \in V_T$.

2) Let $v \neq 0$ be any ultrafunction in the orthogonal (in $\tilde{V}(\Omega)$) of $\tilde{V}(\Omega) \cap \mathcal{D}(\Omega)^*$. Then u and $u + v$ are $\mathcal{D}(\Omega)$ -equivalent, since $\int^* (u(x) + v(x)) \varphi^*(x) dx = \int^* u(x) \varphi^*(x) dx + \int^* v(x) \varphi^*(x) dx = \int^* u(x) \varphi^*(x) dx + 0$. Since the orthogonal of $\tilde{V}(\Omega) \cap \mathcal{D}(\Omega)^*$ is infinite, we obtain the thesis.

□

Remark 31 *There is a natural way to associate a unique ultrafunction to a distribution (see also [1]). In order to do this it is sufficient to split $\tilde{V}(\Omega)$ in two orthogonal component: $\tilde{V}(\Omega) \cap \mathcal{D}(\Omega)^*$ and $\left(\tilde{V}(\Omega) \cap \mathcal{D}(\Omega)^*\right)^\perp$. As we have seen in the proof of the above theorem every ultrafunction in V_T can be splitted in two components, $u + v$ where $v \in \left(\tilde{V}(\Omega) \cap \mathcal{D}(\Omega)^*\right)^\perp$ and $u \in \tilde{V}(\Omega) \cap \mathcal{D}(\Omega)^*$ is univocally determined. Then, we have an injective map*

$$i : \mathcal{D}'(\Omega) \rightarrow \tilde{V}(\Omega)$$

given by $i(T) = u$ where $u \in V_T \cap \mathcal{D}(\Omega)^$.*

Remark 32 *The space of ultrafunctions is richer than the space of distributions; for example consider the function*

$$u(x) := f(x) \min(x^{-2}, \alpha)$$

where $\alpha > 0$ is an infinite number and $f(x)$ is a function with compact support such that $f(0) = 1$. Since $u \in [V'(\Omega) \cap \mathfrak{M}]^$, \tilde{u} is well defined (see def. 25). On the other hand, \tilde{u} does not correspond to any distribution since*

$$\int^* \tilde{u}(x) \varphi^*(x) dx = \int^* f^*(x) \min(x^{-2}, \alpha) \varphi^*(x) dx$$

is infinite when $\varphi(x) \geq 0$ and $\varphi(0) > 0$. In [1] Section 6, it is presented an elliptic problem which has a solution in the space of ultrafunctions, but no solution in the space of distributions.

4 Operations with ultrafunctions

4.1 Extension of operators

Definition 33 *Given the operator $F : V(\Omega) \rightarrow \mathcal{D}'(\Omega)$, the map*

$$\tilde{F} : \tilde{V}(\Omega) \rightarrow \tilde{V}(\Omega)$$

defined by

$$\tilde{F}(u) = \widetilde{F^*(u)} \quad (9)$$

*is called **canonical** extension of F ("~" is defined by 25).*

By the definition of \tilde{F} , we have that

$$\forall v \in \tilde{V}(\Omega), \int^* \tilde{F}(u(x)) v(x) dx = \int^* F^*(u(x)) v(x) dx. \quad (10)$$

Comparing Definition 33 with Theorem 27 we have that

$$\begin{aligned}
\tilde{F}(u(x)) &= \sum_{q \in \Sigma} \langle F^*(u), \delta_q \rangle \sigma_q(x) = \\
&= \sum_{q \in \Sigma} \langle F^*(u), \theta_q \rangle \theta_q(x) = \\
&= \sum_{q \in \Sigma} \langle F^*(u), \sigma_q \rangle \delta_q(x).
\end{aligned}$$

In particular, if $F : V(\Omega) \rightarrow V'(\Omega) \cap \mathfrak{M}^*$:

$$\begin{aligned}
\tilde{F}(u(x)) &= \sum_{q \in \Sigma} \left[\int F^*(u(\xi)) \delta_q(\xi) d\xi \right] \sigma_q(x) = \\
&= \sum_{q \in \Sigma} \left[\int F^*(u(\xi)) \theta_q(\xi) d\xi \right] \theta_q(x) = \\
&= \sum_{q \in \Sigma} \left[\int F^*(u(\xi)) \sigma_q(\xi) d\xi \right] \delta_q(x).
\end{aligned} \tag{11}$$

4.2 Derivative

A good generating space to define the derivative of an ultrafunction is the following one:

$$V^1(\Omega) = H^{1,1}(\Omega) \cap \mathcal{C}(\overline{\Omega}) \subseteq L^2(\Omega) \cap \mathcal{C}(\overline{\Omega}).$$

In order to simplify the exposition, we will assume that $\Omega \subseteq \mathbb{R}$. The generalization of the notions exposed in this section is immediate.

Let $u \in \widetilde{V}^1(\Omega)$ be a ultrafunction. Since $V^1(\Omega)^* \subset H^1(\Omega)^*$, we have that the derivative $\frac{du}{dx} = \partial u = u'$ is in $L^2(\Omega) \subset [V_G' \cap \mathfrak{M}]^*$. Then we can apply Definition 33:

Definition 34 *We set*

$$Du = \tilde{\partial}u = \tilde{\partial}u.$$

The operator

$$D : \widetilde{V}^1(\Omega) \rightarrow \widetilde{V}^1(\Omega)$$

is called (generalized) derivative of the ultrafunction u .

By (11) we have the following representation of the derivative:

$$\forall u \in \widetilde{V}^1(\Omega), \quad Du(x) = \sum_{q \in \Sigma} \left[\int^* u'(\xi) \delta_q(\xi) d\xi \right] \sigma_q(x).$$

If $u' \in \widetilde{V}^1(\Omega) \subset [V^1(\Omega)]^*$, we have that

$$Du(x) = \sum_{q \in \Sigma} u'(q) \sigma_q(x) = u'(x).$$

In particular, if $u \in H^{2,1}(\Omega) \cap \mathcal{C}^1(\overline{\Omega})$, $Du = u'$ and so D extends the operator $\frac{d}{dx} : H^{2,1}(\Omega) \cap \mathcal{C}^1(\overline{\Omega}) \rightarrow V^1(\Omega)$ to the operator $D : \widetilde{V}^1(\Omega) \rightarrow \widetilde{V}^1(\Omega)$.

4.3 Fourier transform

In this section we will investigate the extension of the one-dimensional Fourier transform. A good space to work with the Fourier transform is the space

$$V^{\mathfrak{F}}(\mathbb{R}) = H^1(\mathbb{R}) \cap L^2(\mathbb{R}, |x|^2).$$

It is easy to see that the space $V^{\mathfrak{F}}(\mathbb{R})$ can be characterized as follows:

$$V^{\mathfrak{F}}(\mathbb{R}) = \{u \in H^1(\mathbb{R}) : \hat{u} \in H^1(\mathbb{R})\}.$$

In fact, if $\hat{u} \in H^1(\mathbb{R})$, then $\int |\nabla u(\xi)|^2 d\xi < +\infty$ and hence $\int |u(x)|^2 |x|^2 dx < +\infty$.

Then $V^{\mathfrak{F}}(\mathbb{R}) \subset L^2(\mathbb{R}, |x|^2)$, so $V^{\mathfrak{F}}(\mathbb{R}) \subset H^1(\mathbb{R}) \cap L^2(\mathbb{R}, |x|^2)$ which is a Hilbert space equipped with the norm

$$\|u\|_{V^{\mathfrak{F}}(\mathbb{R})}^2 = \int |u(x)|^2 |x|^2 dx + \int |\hat{u}(\xi)|^2 |\xi|^2 d\xi.$$

Moreover

$$\begin{aligned} \int |u(x)| dx &= \int |u(x)| (1 + |x|) \frac{1}{1 + |x|} dx \leq \\ &\leq \left(\int |u(x)|^2 (1 + |x|)^2 dx \right)^{\frac{1}{2}} \left(\int \frac{1}{(1 + |x|)^2} dx \right)^{\frac{1}{2}} \leq \\ &\leq \text{const.} \left(\|u\|_{L^2(\mathbb{R})} + \|u\|_{L^2(\mathbb{R}, |x|^2)} \right). \end{aligned}$$

Thus, $V^{\mathfrak{F}}(\mathbb{R}) \subset L^1(\mathbb{R})$. Recalling that the functions in $H^1(\mathbb{R})$ are continuous, we have that

$$V^{\mathfrak{F}}(\mathbb{R}) \subset \mathcal{C}(\mathbb{R}) \cap H^1(\mathbb{R}) \cap L^1(\mathbb{R}) \cap L^2(\mathbb{R}, |x|^2).$$

We use the following definitions of Fourier transform: if $u \in \widetilde{V}^{\mathfrak{F}}(\mathbb{R})$, we set

$$\mathfrak{F}(u)(k) = \hat{u}(k) = \frac{1}{\sqrt{2\pi}} \int^* u(x) e^{-ikx} dx; \quad (12)$$

$$\mathfrak{F}^{-1}(u)(x) = \frac{1}{\sqrt{2\pi}} \int^* \hat{u}(k) e^{ikx} dx. \quad (13)$$

Now, in order to deal with the Fourier transform in an easier way, we need a new axiom whose consistency is easy to be verified (see Appendix):

Axiom 35 (FTA) (Fourier transform axiom) If $u \in \widetilde{V}^{\mathfrak{F}}(\mathbb{R})$ then $\mathfrak{F}^*(u) \in \widetilde{V}^{\mathfrak{F}}(\mathbb{R})$ and $\bar{u} \in \widetilde{V}^{\mathfrak{F}}(\mathbb{R})$ (here \bar{u} is the complex conjugate of u).

It is immediate to see that, by this axiom, for every ultrafunction, u we have

$$\mathfrak{F}^*(u) = \widetilde{\mathfrak{F}}(u)$$

and hence, since there is no risk of ambiguity, we will simply write $\mathfrak{F}(u)$.

It is well known that in the theory of tempered distributions we have that:

$$\mathfrak{F}(\delta_a) = \frac{e^{-iak}}{\sqrt{2\pi}};$$

$$\mathfrak{F}\left(\frac{e^{iax}}{\sqrt{2\pi}}\right) = \delta_a.$$

In the theory of ultrafunctions an analogous result holds:

Proposition 36 *We have that:*

1. $\mathfrak{F}\left(\frac{e^{iax}}{\sqrt{2\pi}}\right) = \delta_a(k);$
2. $\mathfrak{F}(\delta_a(x)) = \frac{e^{-iak}}{\sqrt{2\pi}};$
3. $\frac{1}{2\pi} \int^* \widetilde{e^{-iax}} \widetilde{e^{ikx}} dx = \delta_a(k).$

Proof. 1. For every $v \in V^{\mathfrak{F}}$,

$$\begin{aligned} \int^* \mathfrak{F}\left(\frac{e^{iax}}{\sqrt{2\pi}}\right) v(k) dk &= \int^* \left(\frac{1}{2\pi} \int^* \widetilde{e^{-iak}} \widetilde{e^{ixk}} dx\right) v(k) dk = \\ &= \frac{1}{2\pi} \int^* \int^* \widetilde{e^{-iak}} \widetilde{e^{ixk}} v(k) dk dx = \\ &= \frac{1}{\sqrt{2\pi}} \int^* \widetilde{e^{-iak}} \mathfrak{F}^{-1}(v(k)) dx = v(a). \end{aligned}$$

Hence, 1 holds.

2 - We have

$$\mathfrak{F}(\delta_a(x)) = \int^* \delta_a(x) e^{-ikx} dx = \int^* \delta_a(x) \widetilde{e^{-ikx}} dx = \widetilde{e^{-ika}}.$$

3 - We have

$$\frac{1}{2\pi} \int^* \widetilde{e^{iax}} \widetilde{e^{-ikx}} dx = \frac{1}{2\pi} \int^* \widetilde{e^{iax}} e^{-ikx} dx = \mathfrak{F}\left(\frac{e^{iax}}{\sqrt{2\pi}}\right) = \delta_a(k).$$

□

By our definitions we have that:

$$\begin{aligned} \widetilde{e^{ikx}} &= \sum_{q \in \Sigma} \left[\int^* e^{ik\xi} \delta_q(\xi) d\xi \right] \sigma_q(x); \\ \widetilde{e^{ixk}} &= \sum_{q \in \Sigma} \left[\int^* e^{ix\xi} \delta_q(\xi) d\xi \right] \sigma_q(k). \end{aligned}$$

Therefore it is not evident whether $\widetilde{e^{ikx}} = \widetilde{e^{ixk}}$ or not. The following Corollary answers this question.

Corollary 37 *We have that:*

$$\widetilde{e^{ikx}} = \widetilde{e^{ixk}}.$$

Proof. By the previous proposition, we have that

$$\widetilde{e^{-ikx}} = \sqrt{2\pi} \mathfrak{F}(\delta_k(x)) = \int^* \delta_k(x) e^{-ixk} dk = \int^* \delta_x(k) e^{-ixk} dx = \widetilde{e^{-ixk}}.$$

Replacing x with $-x$ we get the result.

□

Since $\mathfrak{F}: V^{\mathfrak{F}}(\mathbb{R}) \rightarrow V^{\mathfrak{F}}(\mathbb{R})$ is an isomorphism, it follows that, for any Delta-basis $\{\delta_a\}_{a \in \Sigma}$, the set

$$\left\{ \frac{\widetilde{e^{iax}}}{\sqrt{2\pi}} \right\}_{a \in \Sigma} = \{\mathfrak{F}(\delta_{-a})\}_{a \in \Sigma}$$

is a basis and we get the following result:

Theorem 38 *If $u \in V^{\mathfrak{F}}(\mathbb{R})$, then*

$$u(x) = \frac{1}{\sqrt{2\pi}} \sum_{k \in \Sigma} \underline{\underline{\hat{u}}}(k) \widetilde{e^{ikx}}.$$

where we have set (see Theorem 24)

$$\underline{\underline{\hat{u}}}(k) = \int^* \hat{u}(\xi) \sigma_k(\xi) d\xi.$$

Proof. Since $\left\{ \frac{\widetilde{e^{ikx}}}{\sqrt{2\pi}} \right\}_{k \in \Sigma}$ is a basis, any $u \in V^{\mathfrak{F}}(\mathbb{R})$ has the following representation:

$$u(x) = \frac{1}{\sqrt{2\pi}} \sum_{k \in \Sigma} u_k \widetilde{e^{ikx}}.$$

Let us compute the u_k 's: we have

$$\int \delta_k(x) \overline{\sigma_b(x)} dx = \int \delta_k(x) \sigma_b(x) dx = \delta_{kb}$$

and so

$$\int \hat{\delta}_k(x) \overline{\hat{\sigma}_b(x)} dx = \delta_{kb}$$

and by Proposition 36,

$$\int \frac{\widetilde{e^{-ikx}}}{\sqrt{2\pi}} \overline{\hat{\sigma}_b(x)} dx = \delta_{kb}.$$

Hence $\{\widehat{\sigma}_k(x)\}_{k \in \Sigma}$ is the dual basis of $\left\{\widetilde{\frac{e^{-ikx}}{\sqrt{2\pi}}}\right\}_{k \in \Sigma}$, namely $\{\widehat{\sigma}_k(-x)\}_{k \in \Sigma}$ is the dual basis of $\left\{\frac{e^{ikx}}{\sqrt{2\pi}}\right\}_{k \in \Sigma}$. Hence, since $\widehat{v}(x) = v(-x)$, we have:

$$\begin{aligned} u_k &= \int u(\xi) \overline{\widehat{\sigma}_k(-\xi)} d\xi = \int u(\xi) \overline{\widehat{\widehat{\sigma}_k(-\xi)}} d\xi = \\ &= \int \widehat{u}(\xi) \overline{\sigma_k(\xi)} d\xi = \int \widehat{u}(\xi) \sigma_k(\xi) d\xi = \underline{\underline{u}}(k). \end{aligned}$$

□

5 Appendix

In this section we prove that the axiomatic construction of ultrafunctions is coherent. We assume that the reader knows the key concepts in Nonstandard Analysis (see e.g. [7]).

The following result has already been proved in [1]. Here we give an alternative proof of this result based on Nonstandard Analysis:

Theorem 39 *The set of axioms $\{(\Lambda-1), (\Lambda-2), (\Lambda-3)\}$ is consistent.*

Proof. Let \mathbb{U}, \mathbb{V} be mathematical universes and let $\langle \mathbb{U}, \mathbb{V}, \star \rangle$ be a nonstandard extension of \mathbb{U} that is $|\mathbb{U}|^+$ -saturated. We denote by \mathcal{F} the set of finite subsets of \mathbb{U} and, for every $\lambda \in \mathcal{F}$, we pose

$$F_\lambda = \{S \subset \mathbb{V} \mid S \text{ is hyperfinite and } \lambda^* \subset S\}.$$

By saturation $\bigcap_{\lambda \in \mathcal{F}} F_\lambda \neq \emptyset$. We take $\Lambda \in \bigcap_{\lambda \in \mathcal{F}} F_\lambda$.

For any given net $\varphi : \mathcal{F} \rightarrow \mathbb{U}$ we define its Λ -limit as

$$\lim_{\lambda \uparrow \mathbb{U}} \varphi(\lambda) = \varphi^*(\Lambda)$$

and we pose

$$\mathbb{K} = \lim_{\lambda \uparrow \mathbb{U}} \mathbb{R} = \left\{ \lim_{\lambda \uparrow \mathbb{U}} \varphi(\lambda) \mid \varphi : \mathcal{F} \rightarrow \mathbb{R} \right\}.$$

With these choices the Λ -limit satisfies the axioms $(\Lambda-1), (\Lambda-2), (\Lambda-3)$: the only nontrivial fact is $(\Lambda-2)$. Let φ be an eventually constant net, and let $\lambda_0 \in \mathcal{F}, r \in \mathbb{R}$ be such that $\forall \lambda \in \{\eta \in \mathcal{F} \mid \lambda_0 \subset \eta\}$

$$\varphi(\lambda) = r.$$

By transfer it follows that $\forall \lambda \in \{\eta \in \mathcal{F} \mid \lambda_0 \subset \eta\}^* = \{\eta \in \mathcal{F}^* \mid \lambda_0^* \subset \eta\}$ we have:

$$\varphi^*(\lambda) = r^*.$$

But $r = r^*$ and $\lambda_0^* \subset \Lambda$ by construction. So, since $\Lambda \in \mathcal{F}^*$, $\varphi^*(\Lambda) = r$.

□

Having defined the Λ -limit, from now on we use the symbol $*$ to denote the extensions of objects in \mathbb{U} in the sense of Λ -limit (not to be confused with the extensions obtained by applying the star map \star : e.g., the field $\mathbb{K} = \mathbb{R}^*$ is a subfield of \mathbb{R}^*).

We observe that, given a set S in \mathbb{U} , its hyperfinite extension (in the sense of the Λ -limit) is

$$S^\circ = \lim_{\lambda \uparrow \mathbb{U}} (S \cap \lambda) = S^* \cap \Lambda$$

and we use this observation to prove that, given a set of functions $V(\Omega)$, by posing

$$\tilde{V}(\Omega) = \text{Span}(V(\Omega)^\circ) = \text{Span}(V(\Omega)^* \cap \Lambda)$$

we obtain the set of ultrafunctions generated by $V(\Omega)$.

The only nontrivial fact to prove is that, for every function $f \in V(\Omega)$, its natural extension f^* is an ultrafunction. First of all, we observe that, by definition, $f^* = f^*$. Also, since $f \in V(\Omega)$, by transfer it follows that $f^* \in V(\Omega)^*$. And, by our choice of Λ , we also have that $f^* \in \Lambda$ since, by construction, $\{f\}^* = \{f^*\} \subset \Lambda$.

It remains to prove the coherence of the axioms $(\Lambda-1), (\Lambda-2), (\Lambda-3)$ combined with FTA .

Theorem 40 *The set of axioms $\{(\Lambda-1), (\Lambda-2), (\Lambda-3), FTA\}$ is consistent.*

Proof. The basic idea is to chose an hyperfinite set $\Lambda \in \bigcap_{\lambda \in \mathcal{F}} F_\lambda$, where F_λ is defined in Theorem 39 (which automatically ensures the satisfaction of $(\Lambda-1), (\Lambda-2), (\Lambda-3)$), with one more particular property that will ensure the satisfaction of FTA .

We start by considering a generic hyperfinite set $\Lambda' \in \bigcap_{\lambda \in \mathcal{F}} F_\lambda$ and we let

$$B' = \{e_i(x) | i \in I\}$$

be any hyperfinite basis for $\text{Span}(V^{\mathfrak{F}}(\mathbb{R})^* \cap \Lambda')$. Now we pose

$$B = \{\mathfrak{F}^j(e_i(x)) : 0 \leq j \leq 3, i \in I\} \cup \{\overline{\mathfrak{F}^j(e_i(x))} : 0 \leq j \leq 3, i \in I\},$$

where \mathfrak{F} denotes the Fourier transform. Since $\mathfrak{F}^4 = id$, we have that B is closed by Fourier transform and complex conjugate. We now pose

$$\Lambda = \Lambda' \cup B$$

and it is immediate to prove that, with this choice, FTA is ensured, because B is a set of generators for $\widetilde{V^{\mathfrak{F}}(\mathbb{R})}$ closed by Fourier transform and complex conjugate.

□

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